PHYSICAL-MATHEMATICAL CONCEPTS IN A SIMPLIFIED STUDY OF THE PROTOCOL OF THE LAUNCH OF A ROCKET TO PUT A SATELLITE INTO ORBIT

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Abstract
The achievement of a comprehensive education of the students to provide a global knowledge that allows them to make appropriate decisions in the framework of their future work is one of the most important challenges of the Engineering degrees, especially in basic subjects as Physics or Mathematics. In this sense, a great effort has to be done by the teachers of the first year where the subjects are generally quite abstract and, as a consequence, the motivation of the students is hard to attain. Several are the conditions to obtain this goal successfully: First, a correct cross-coordination and a well-designed teaching of the basis subjects to avoid undesirable repetition of some concepts under different point of view and to show the basic knowledge as a uniform conceptual body ready to be applied to real situations. Second, the need to make attractive the study of these basic and abstract subjects by proposing the resolution of real problems related with the goals of each particular Degree, and adapted to the knowledge of the student of the first level. These problems have to be solved involving different branch of the basic subjects as Physics or Mathematics.

In this paper we present a real project related to the professional skills of the Geomatic and Topography Degree, adapted for the first level, and involving three core subjects namely, Algebra, Calculus and Mechanics. The approach of this real project, proposed as an interdisciplinary practice, is related with the path followed by a rocket that puts a satellite in orbit. In real situations, sixteen are the steps of the protocol to achieve this goal, from the takeoff of the rocket to its entrance on the final elliptic orbit.

To adapt this complicated protocol to our purposes, we have simplified the steps considering finally only four: (i) the takeoff of the rocket; (ii) the change of the upright path at a predetermined height from the ground to approach to the elliptical orbit; (iii) the link between the rise of the rocket and its placing in elliptical orbit; (iv) the putting into the elliptical orbit itself.

Analysis of classic geometric figures as the conics, with their different forms of representation, becomes necessary to be able to develop this practice, since they adapt to the forms described by orbits of the satellites. Modeling makes it necessary to use different reference systems and their relationship between them. In addition, the simplified study of the satellite position vector in its different phases provides fundamental data such as speed, acceleration or balance of forces, related with the subjects of Dynamic or Kinematics, and where the concept of derivative plays a fundamental role. A later detailed study on the motion of the satellite would lead to propose systems of differential equations that relate the elements studied and whose complexity requires numerical methods for their resolution.

Keywords: Multidisciplinary learning, Theoretical-experimental learning.

1 INTRODUCTION
Comprehensive education of students is one of the most important challenges in the different degrees of Engineering. In a working world where the technical solutions derive from many -and sometimes complicated- subjects, a global and well-structured knowledge provides student the right approach to each engineering problem, however different they may be. But the development of this ability is not
easy. To achieve this goal, a proper coordination of the different subjects that form the complete curricula is needed. The key in this process is the teaching of the basic subjects, as Mathematics or Physics, since they are the basis of the engineering knowledge. Thus, the creation of a strong foundation of this global knowledge goes through a correct cross-coordination of these basic subjects. Moreover, this coordination provides positive effects on the learning process of students, attracting their interest due to a more global problem-solving approach, reducing the dropout rate, avoiding undesirable repetition of some definitions or concepts and contributing to the establishment of a strong scientific baseline to address engineering problems.

Following this line, in this paper we present a coordinated practice to develop in the first year of the Geomatic and Topography Engineering Degree, involving three basic subjects as Algebra, Calculus and Mechanics. The proposed practice is based on the Physical calculations contents in the real protocol of the launch of a rocket to put a satellite into orbit. Obviously, this real protocol is very complex and has a high technical knowledge level, far away from the one that the students have in a first course of an engineering degree. A complete development of this problem can be found in [1]. However, the conceptual basis of the protocol can be easily explained using some elementary arguments related with the notion of motion—which is a key concept in the subject of “Mechanics”—, and the resulting equations can be solved easily using mathematical techniques taught in the first course of the engineering degrees. Therefore, the conceptual modelling of the real case to adapt the protocol of the launch to the scientific level of the first year of an engineering degree is the most important step in the development of the practice. Throughout the solving process, an appropriate mathematical software (Mathematica) will be used to help students in the calculations.

The real standard protocol considered involves four steps, which includes up to sixteen phases, from launching the rocket to putting the satellite into orbit. Also, all this start-up process of the protocol includes some preliminary important decisions about both the starting conditions of the problem or the choice of a coordinate system. In next sections we define and develop the modelling of the protocol fixing its starting conditions, including the questions that have to be solved by the students. These questions are related with the motion of the rocket and the satellite. Thus, in section 2 we describe both the modelling of the rocket launch and the putting into orbit of the satellite, and the Mechanics modelling and the Physical questions related with the motion of the device. Section 3 shows the mathematical modelling of the considered problem and the Algebra and Calculus questions related with it. Finally, section 4 summarizes the conclusions of the presented practice.

2 PHYSICAL MODELING OF THE PROTOCOL OF PUTTING A SATELLITE INTO ORBIT

The main step in the development of this practice is the physical modelling of the real and complex protocol, in such a way the problem can be solved by the students. In this section we determine the assumptions considered to simplify the real protocol using the physical knowledge adapted to the first course of the Geomatics and Topography Degree at the Polytechnic University of Valencia. In this Degree, students have a subject called “Mechanics” related with the analysis of the motion of bodies. In this subject, we analyse two important parts. On one hand, the geometry of the motion of the bodies without considering the forces that produce it is analysed by the kinematics. Specifically, in this part of Mechanics, we analyse briefly the main parameters that describe the geometry of the motion, i.e. position, velocity and acceleration vectors applied to the case of a particle or a body that describes a plane motion. On the other hand, dynamics analysed the relationship between the geometry of the motion and the forces that produce the motion. In this part of Mechanics, students analysed again the case of a particle and a body that follow a plane motion. Here, we develop two method of analysis: the first one is based in the concept of force and is based on the Newton’ laws and the second on the concepts of work and energy. Finally, one of the lessons is focused on the concept of gravitation, where the Kepler’ laws are explained and the application of the dynamic resolution methods to the case of the motion of bodies on the space is done.

In this practice we focus our attention in the case of kinematics, where the students have to work with the geometry of the motion, and have to obtain some expressions related to the position, velocity and acceleration vectors. In order to obtain only one expression for each of these vectors, we will write their expressions as functions of variable scalar the time.

The proposed problem is related to the putting into orbit of a satellite. In this problem we have considered some general restrictions: (i) the satellite describes an elliptical orbit. The perigee and the apogee of the orbit are 358 km and 2550 km from the surface of the Earth respectively; (ii) the rocket
takes off from a point on the equator; (iii) we consider our modelling as a problem of plane motion and (iv) the acceleration of gravity is 9.8 m/s² and the acceleration of the rocket engines should be defined for each case.

Taking into account this background, we have fixed the starting conditions of the problem. Both the general scheme of the modelling and the main parameters used in the calculations are described in Fig. 1a. In this figure we define first a fixed reference system (OXeYeZe), which centre is located in the centre of the Earth (point O), and with respect to which the Earth rotates with an angular velocity of modulus \( \omega_{\text{Earth}} \) and direction along the positive Ze axis. Note that, due to the position of the reference system chosen, the height of the rocket/satellite is considered from the centre of the Earth, not from its surface. Due to the consideration of plane motion for the satellite, the elliptical trajectory is contained in the OXeYe plane.

We have also simplified the real protocol of putting a satellite into orbit, transforming it as follows:

(i) In phase 1, the rocket describes a straight path, following a uniformly varied rectilinear motion, with an acceleration formed by the sum of that of the rocket engines and the acceleration of gravity. The estimated time to finish this phase is 155 s, traveling a distance equal to 117 km.

(ii) In phase 2 the same motion conditions are conserved, but the rocket rotates an angle of 40 degrees with respect to the local horizon in the direction of rotation of the earth. In this phase, the rocket travels until 134 km from the surface of the Earth, and the time required to finish this phase is 7s.

(iii) In phase III, the rocket turns off the engines and follows a parabolic trajectory to reach the Apex (700 km from the surface of the Earth). In this phase only the gravity acceleration works and the rocket follows a uniformly varied motion. The time spent in this phase is 271 s.

(iv) Finally, in phase IV, the rocket is placed with an angle of 0 degrees with respect to the local horizon in the direction of rotation of the earth. The time spent in this phase is 6 s. Now, the satellite is putting in his elliptical orbit.

With these conditions, the goal of the practice is to express the position, velocity and acceleration vectors of the satellite, since the take off until it is put into orbit, as vector functions of variable scalar time. These vector functions have the general expressions:

\[
\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \quad m
\]

\[
\vec{v}(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} = v_x(t)\hat{i} + v_y(t)\hat{j} \quad m/s
\]
\[ \ddot{a}(t) = \frac{d^2x(t)}{dt^2} \hat{i} + \frac{d^2y(t)}{dt^2} \hat{j} = v_x(t) \hat{i} + v_y(t) \hat{j} \quad \text{m/s} \quad (3) \]

Note that these functions will have to be expressed as functions in tranches, as in the next expression for the case of the position vector:

\[
\ddot{r}(t) = \begin{cases} 
\text{Phase I: } x_1(t) \hat{i} + y_1(t) \hat{j} & \text{m} \\
\text{Phase II: } x_2(t) \hat{i} + y_2(t) \hat{j} & \text{m} \\
\text{Phase III: } x_3(t) \hat{i} + y_3(t) \hat{j} & \text{m} \\
\text{Phase IV: } x_4(t) \hat{i} + y_4(t) \hat{j} & \text{m} 
\end{cases} \quad (4) \]

3 MATHEMATICAL BASIC KNOWLEDGE

3.1 Algebra

The first concept that must be known to adequately represent the position of an object moving is the system of reference in \( \mathbb{R}^2 \), \( S = \{0, e_1, e_2\} \), or in \( \mathbb{R}^3 \), \( S = \{0, e_1, e_2, e_3\} \). If the movement is in a plane, as in our case, we fix in \( \mathbb{R}^2 \) a reference system giving the relation between cartesian coordinates \((x, y)\) and polar coordinates \((r, \varphi)\) of a point (see Fig. 2), this is: \( x = r \cos \varphi \), \( y = r \sin \varphi \), as well as the reference system change equations from the matrix point of view. It should be noted that, for example, when the position of a satellite is studied and a three-dimensional reference system is used, the relationship between the reference system change matrices is explained by the Euler angles (or their variations, see [1]), and the rotation matrices (in the case of systems with the same orientation) or rotation and symmetry matrices associated with them (when systems have different orientation).

[Figure 2: Relation between cartesian coordinates and polar coordinates.]

Kepler’s laws govern the satellite motion around Earth. The first of these scientific laws establish that the satellite describes a plane orbit, which is an ellipse with a focus on the Earth’s center of mass, sweeping its vector radius equal areas at equal times (that is, it has constant areolar velocity). Thus, following the Kepler’s laws, the orbits are defined from: (1) their shape (circular orbits and elliptical orbits); (2) its orbital plane with respect to the equatorial plane or the \( \theta \) inclination, in degrees, with respect to Ecuador (equatorial orbits, tilt orbits and polar orbits) as we can see in Fig. 3.
Finally, these orbits they are also defined by (3) their distance to Earth (see Fig. 4):

a) **Low Earth Orbit, LEO:** low orbit between 500 km and 1500 km, \( \theta = 90^\circ \);

b) **Medium Earth Orbit, MEO:** orbits between 5000 km and 12000 km, \( \theta = 45^\circ \);

c) **Geosynchronous Orbit, GSO:** The orbital period of these orbits is the same than the one followed by the sidereal rotation of Earth, describing an elliptical orbital with semi-major axis \( a = 42164 \text{ km} \).

d) **Geostationary Orbit, GEO:** is a GSO with circular orbit on the line of the Ecuador, \( \theta = 0^\circ \), being the distance from the Earth of 35794 km approximately.

This last classification takes into account the influence of the so-called Van Allen belt (the first between 1500 km and 3000 km and the second between 13000 km and 20000 km). This belt should be avoided by the satellites due to the high number of particles that forms it (see [2] and [3]). To do that, the orbiting of a satellite in geosynchronous orbit takes place in several steps: first it is taken to a LEO orbit, then it is passed to an elliptical transitional orbit and finally it is taken to the geosynchronous orbit.

On the other hand, in this practice we have to take into account the solution of the gravitational orbit equation: 

\[
\frac{d^2}{d\varphi^2} \left( \frac{1}{r} \right) + \frac{1}{r} = \frac{GM}{c^2},
\]

(a second order linear differential equation in \( \frac{1}{r} \)), which is the general expression of a conic in polar coordinates, 

\[
r = \frac{p'}{1 + ep'}
\]

where \( r \) represents the distance from the conic to the focus, \( \varphi \) is the angle with respect to the focal axis, \( e \) is the eccentricity of the conic and \( p' \) is a value related to the parameter \( p \) of the conic (in this paper \( p' = 2pe \)).

If we consider an energy criterion, 

\[
E = -\frac{1}{2} \frac{GMm}{p'} (1 - e^2),
\]

three cases appear: (i) if the total energy is negative \((e < 1, E < 0)\), the satellite describes a closed (elliptical) orbit; (ii) if the total energy is equal to zero \((e = 1, E = 0)\), the satellite follows a parabolic orbit because it has the right energy to reach infinity disappearing its kinetic energy in the process; (iii) Finally, if the total energy is positive...
\( e > 1, E > 0 \), the satellite describe a hyperbolic path escaping from the Earth’s gravitational attraction due to its kinetics energy is higher.

All these Mechanics arguments suggest the need to study the conics in a matrix form to work with the matrix expressions of reference system change in a parametric form. With this approach it is possible to define properly the position vectors according to the different sections of conic that are joining together in a metric form. Moreover, this matrix approach allows us to give a geometric interpretation of both the definition and classification of the conics formed, and the way in which we can go from one type of conic to another depending on the metric elements that define them.

Thus, the approaching of the problem is to begin with the metric definition of a conic as the locus of points \( X = (x, y) \) whose distances with respect to one given (focus: \( F \)) and to a given line (directrix: \( d \)) are in a constant relationship and equal to \( e \geq 0 \), so:

\[
\frac{\text{dist}(X,F)}{\text{dist}(X,d)} = e \quad \text{and} \quad \begin{cases} \quad e = 0, & \text{Circle} \\ \quad 0 < e < 1, & \text{Ellipse} \\ \quad e = 1, & \text{Parabola} \\ \quad e > 1, & \text{Hyperbola} \end{cases}
\]

In Fig. 5 we see the interpretation of those elements with parameter \( p = \frac{1}{2}d(F, d) \):

![Figure 5: Geometric interpretation of conics.](image)

Therefore, the different types of trajectories can be perfectly interpreted according to the parameters of the conics, starting from the initial definition that we have proposed, since in the definition of elliptical orbit the Earth occupies one of the focus of the ellipse. Fig. 6 shows the elliptical orbits (solid orange lines), the parabolic trajectory that represents the escape path of the satellite (solid red lines) and the hyperbolic paths (solid lines with different colors), considering that the Earth is located in a focus of these conics.
With all these arguments, we propose in this practice to carry out a first study to obtain, from the parameter \( p \) and the eccentricity \( e \), the equations of the conics with the same focus and directrix line, the coordinates of the second focus and the center, as well as the metric properties that characterize them, taking into account that in an ellipse the sum of the distances to the focus is constant and equal to \( 2a \), and in an hyperbola, the module of the difference of the distances to the focus is constant and equal to \( 2a \).

Special attention deserves the elliptical trajectory that is studied. Its parametric equations can be expressed as follows: \[ \begin{align*}
    x &= a \cos u \\
    y &= b \sin u,
\end{align*} \] where the parameter \( u \), called eccentric anomaly, is related to the angle \( \varphi \), true anomaly, which is the angle that forms the radius vector with the focus \( F \) (see Fig. 8). The relationship between them is \( \cos u = e + \frac{r}{a} \cos \varphi \). This last expression together with the property of the areolar velocity allow the obtaining of the formula \( u - e \sin u = \frac{2\pi}{T} t \), which is useful to obtain the position \( r \) of the satellite according to the corresponding eccentric anomaly. Similar studies can be done for parabolic and hyperbolic trajectories.
3.2 Calculus

Regarding the subject of Calculus, we would like to highlight the concepts and results related to differential and integral calculus that could appear in this practice. Mainly we focus our attention on the concept (i) of derivative for functions of a variable, (ii) of partial derivative for functions of several variables, and (iii) the meaning and application of integration, where we have considered, for convenience in this subject, the velocity and acceleration, as scalar magnitudes.

In order to the better understanding of the meaning of the velocity and acceleration concepts, \( v(t) = \frac{dr}{dt}, a(t) = \frac{dv}{dt} \), we propose the construction of tables with specific data to practice with the concept of incremental ratios \( \frac{r(t+\Delta t) - r(t)}{\Delta t} \) and \( \frac{v(t+\Delta t) - v(t)}{\Delta t} \) or the equivalent of the latter ratios \( \frac{r'(t+\Delta t) - r'(t)}{\Delta t} \). This ratio is more difficult to understand, as the same derivative appears in the quotient.

It is convenient in the development of the practice the analysis, individually or in groups and through some examples, of the previous ratios by means of different tables using the software Mathematica. Thus, for each given time, students have to calculate different values of each term of the previous expressions. This will be convenient to do with specific previous physical examples taught in Physics.

The above table-based approach allows the students to observe how approximate is the value obtained to the exact value of the derivative at a given time. Note that this real value is usually calculated in Mechanics, but this procedure is not really understood by the students. Along with this it agrees with the software used, to simulate geometrically and in movement with the command of the program used, how the secant lines tend, given a concrete value of the time and with different increments, to a limit position that is the tangent line, when the increase in time tends to zero. The tangent of the angle that forms that line with the axis OX is what must be transmitted to the student that must coincide with the values to which the table of the incremental quotient given before, is the value of the derivative in the point, both for space as for velocity.

As a second part of the Calculus part of the practice we consider how to transmit to the student the idea of modeling a physical phenomenon through mathematical formulation. To do that, we start from a classic problem of Physics related to the fall of a body vertically by the action of gravity, starting from the rest. Students have to interpret the conditions associated with the motion, related to the concept of derivative. After discussion we propose a simple differential equation \( v'(t) = g \), with the initial condition \( v(0) = 0 \) to model this motion.
In the third section of the calculus part of the practice, we can make some considerations regarding to the launch of a rocket to put a satellite into a particular orbit taking into account only two forces: the gravity of the Earth and the one due to the engines of the rocket, as we have established in section 2. Thinking about the orbit of the satellite and taking into account the laws of Kepler (see [3], [4]), we can implement the equations of location of the satellite into the orbit (see [1]) relating with the concept of derivative in the software used.

Finally, it would be convenient to have in our approach a formulation to simplify situations in the problem of the putting a satellite into orbit and that can show the student clearly the joint use of the derivative and the integral. To do that, we could consider an exercise based on data similar to the following.

We consider as \( m_i \) the initial mass of a rocket that leaves the earth, \( c \) the energy consumption per second with a constant velocity of \( k \) cm / sg relative to the rocket and we assume that no external force acts on the rocket. Of course the mass after \( t \) seconds will be \( m(t) = m_i - ct \), ie the initial mass minus the fuel consumption in \( t \) seconds. The velocity of the fuel gas relative to the rocket is \( v = -k \), considered constant, and \( V(t) \) is the velocity of the rocket relative to the Earth. Considering the equality \( m(t)V'(t) - vm'(t) = 0 \) (see [4]) that relates the previous data, the relationship is reached which motivates in the student the desire to know \( V(t) \) and this allow us to justify to the student the need to integrate interpreting initial conditions. Then we can discuss and implement also the distance covered at a time \( t_k \) elapsed, starting from the surface of the Earth, which it will be sufficient to make use of \( V(t_k) = \frac{dr}{dt}(t_k) \) and integrate again. Certain constraints limiting the theoretical limit of flight time and they must be taken into account in the above equations.

We want to add that in the part of the subject of Algebra we approach parametric equations such as ellipses in relation to elliptic orbits, and this can also be used to combine exercises on derivation in this type of equations.

4 CONCLUSIONS

All the previous approach, based on the modeling of real situations, must lead to a greater implication of students in the acquisition of new concepts and a global and well-structured knowledge that forms a strong foundation based on basic subjects as Mechanics, Algebra or Calculus. The relationships between these subjects, usually studied as separated containers of concepts, allow students to clarify and to join this basic knowledge searched. In this process, it seems very important the support material provided to the student by means of internet videos and files that are on the platform used in the Universitat Politècnica de València to exchange files, called PoliformaT. The joining of all these parts allows a combined methodology, where flipped classroom plays an important role, to consolidate the global knowledge searched. Taken into account that the student works usually in groups, we think that there will be a good disposition of the student as it has happened in other occasions where we have put forward joint practices as presented in [5]. Thus, this kind of experiences provides a better adaptation in the acquisition of transversal competences that are worked on the involved subjects, such as teamwork and leadership and Analysis and problem solving.

REFERENCES