AN ORIGAMI MODEL TO INVESTIGATE SQUARES: FROM GEOMETRY TO FRACTALS

U. Zich
Politecnico di Torino (ITALY)

Abstract
This paper describes the idea to design an origami in order to explain the scale ratio 1:2 and the activities with it to develop geometric skills when folding a square it is possible to introduce reasoning related to linear and surface relationships. Many origami models allow to "see", in its multiple transformations during the folding sequence, polygons; in this case the sequence, which leads to the modelling of a 2D Fractal model, allows us to introduce not only the recognition of plane figures but also the concept of passage of scale, to become familiar with the concept of fractal, to observe fractions and visualize equivalent areas.

Keywords: Scale ratio, fractions, origami, fractal, geometric sequence.

1 INTRODUCTION
There are a lot of shared educational origami experiences to teach geometry in primary school – "in paper-folding several important geometric process can be effected much more easily than with a pair of compasses and ruler, the only instruments the use of which is sanctioned in Euclidean geometry" [1] - and the one I will propose here is a co-operation between math and representation in the metacognitive process of concrete understanding of geometry [2]. Origami, although transversal to many disciplinary sectors, has properties that make it particularly effective in the Mathematics / Representation relationship, becoming a privileged laboratory for learning geometry [3]. His being at the same time a tactile and visual experience, initially guided and then rapidly autonomous, instantaneous and infinite - in the sense that it lives by the emotion of the moment of its creation but also remains a repeatable product and therefore always renewable. This makes it an effective tool for teaching mathematics in primary schools - and not only, as widely shared by the literature, [4], [5] – and other disciplines that use geometry as a language or visualization tool - art, image and geography-. The scale of representation is a subject of the geography discipline, in the third or fourth class of primary school: it is normally presented only as an aspect linked to the representation of maps but also it is in direct relation with fractions. The fact of linking them and using the proposed experience to introduce reasoning related to linear and surface relationships makes it possible to realize a series of transversal applications that educate the student to make use of more skills and exit from the scheme that sees these skills related to the single discipline and not part of a wider culture. Furthermore, the finished model becomes an object that can decorated with the management of the layers of paper that create ‘invisible’ decomposed figures.

2 METHODOLOGY
The critical rereading, between text and images, of Montessori’s material geometry, suggested me, at first, to use origami to try to visualize some of the exercises described in the text and, in a second moment, to "prepare an eloquent object in its mute content, on the periphery of the mind to link it with interest to the object. The eloquence of the mute object will be like a secret revealed to those who projected their intellectual energy on it. To reach this result, the process is different from the usual one: it is not setting the thought above an idea: it is reworking an object, and so retaining it before the senses, making it move with continuous movements, reproducing it with sensitive images." [6]. These considerations underline the importance of the manipulation phase as an educational moment. Hence the idea to design an origami in order to explain the 1:2 scale ratio in order to develop geometric skills. Essential requirements for this project were:

- easy folding to reproduce the model quickly in an autonomous way and assimilate its geometric peculiarities;
- dynamism of the model because the open/closed interaction allows to pass from one scale factor to another and therefore to compare quantities to understand them better.
2.1 A manageable material ... to induce the mind to think

M. Montessori describes, in her *Psychogeometry* [6], a series of activities that involve the use of cardboard figures to discover shapes, defined by “contours” and “values”, that are perimeters and areas. The use of these materials makes it possible to read by hand what appears distant if read on a book. In this perspective, the origami model suits well to functional learning exploration: it offers hand/mind integration in a simple and intuitive way because it allows to “see”, in its multiple transformations during the folding sequence, many concepts that from theoretical become concrete ones.

For a comparison between static images (often inserted by the author at the end of the text and sometimes placed between the lines in the following editing of her works), used to describe a dynamic activity, and semidynamic images, such as a snapshot of a dynamic sequence which leads to the finished model, it is necessary to remember the different nature of the materials used: in one case, rigid physical models, not transformable, in the other paper, material certainly subject to transformation.

Below are some Figures inserted in the Montessori text and their origami repetition.

2.1.1 The subdivision of the square

M. Montessori, describing the subdivision of the square, says that “it is whole and then subsequently divided into two, four, eight and sixteen parts, in two series, that is: by means of diagonals, in ever smaller triangles; and by means of medians in quadrangular forms” (p.79, [6]).
When it enriches the description - "let us first consider the square divided into two triangles. The two triangles match along a line that going from a vertex of an angle to the opposite one, divides all the shape of the square. This line is called the diagonal [...] the two diagonals of the square divide it into four equal triangles [...] as can be seen by comparing and superimposing them. "[6] - it uses a geometric, technical language, without introducing the concept of symmetry. Using origami, to explain the sequence of folds to children, we instead often use the concept of symmetry precisely because the paper allows us to easily verify it by bringing the semi-surfaces in contact and the use of two-colored paper highlights it.

2.1.2 Construction of equivalent figures

From the comparison in Fig. 1 emerge the intrinsic potentialities of the materials that partly condition what becomes more immediately verified with hand. Also for the construction of equivalent figures the choice of materials becomes fundamental: the use of montessori’s cards allows to overlap the plane figures and verify their geometric peculiarities, which is also possible with the origami using dedicated models and being able to approach or overlap them easily. With a few folds it is possible to visualize the series 1/2, 1/4, 1/8, 1/16, both in the construction with the medians and in the one with the diagonals and therefore understanding that the value of two different forms can still be the same.

Figure 2. Given Square Q, with side l, two representations of area ½ are shown from left (a-b); on the right (c-d), the same dynamic is managed as a series and allows the ratios 1/2, 1/4, 1/8, 1/16 to be displayed.

2.2 Design the model: from the idea to the folding sequence

With the intention of observing the aforementioned proportional relationships with a single paper support, a rigorous project is required to move the areas that are intended to be compared and to hide the excess paper through folding (and overturning) operations. Therefore, in order to design an origami with the aim of investigating the 1:2 scale ratio, it was necessary to identify the right model in order to be able to simultaneously see the two areas to be compared and make the bending process reversible. “The possibility of going back a few steps in the process, makes Origami an exploration medium similar to software virtual environments. Unlike the latter ones, Origami requires the student’s active involvement in manipulating the material.” [4] The dynamism of the designed model makes it possible to become familiar, in an informal way, with fractions and equivalent areas, directly observing many geometric transformations. The choice of the model, its design and even the very sequence of folding becomes an opportunity for research in the intention of making it more acceptable. In this case, the folding sequence has been optimized to make it accessible and repeatable in an autonomous way for primary school students.

2.2.1 Sequence of folding and observation of the model in progress

The folding sequence has been optimized to make it accessible and repeatable in an autonomous way for primary school students. The following description is the one used in the IV classes where the experimentation was done. The language chosen is geometric and synthetic; not to be confused, it uses expressions that are proposed in the same way, at different times, in identical passages

a) Starting from a square Q, edge l and area $l^2$, choose a paper differently coloured in the two sides. Referring to figure 3, we will call these sides: green and orange.

b) Using the green Q area, you have to identify the midpoint of two opposite sides and display the line that connect them: this is the median. The median divides the square into two identical rectangles; it is an axis of symmetry of the square. Let us verify this by folding the median and finding a perfect correspondence between the parts.
c) Now you see an orange rectangle with two layers of paper. Identify the longest median of the rectangle of the top layer of paper and fold it.

d) Now you see two green rectangles.

e) Unfold the model. What can you see? The green square $Q$ divided into a large rectangle, equal to $Q/2$ and two smaller rectangles each equal to half of the large rectangle. In this passage it is possible to see the whole, the half and the quarter of the $Q$ square using dissimilar quadrilaterals.

f) Repeat step b-d with the other edge in the perpendicular direction.

g) What can you see on the green side? Two rectangles of area $Q/8$ and 4 squares of area $Q/16$. 

Figure 3. Sequence of folding.

$Q = \text{square, edge } l, \text{ area } l^2$

$A_0 = \text{square, edge } l/2, \text{ area } l^2/4$

$B_0 = \text{square, edge } l/4, \text{ area } A_0/4 \text{ or } l^2/16$
h) Unfold the model. What can you see? The two medians of the square could have divided the initial square into four squares. Area \(Q/4\). The presence of other two foldings highlights 9 quadrilaterals: a square area \(Q/4\), four rectangles area \(Q/8\) and four squares area \(Q/16\). In this way it is possible to observe a series of possible combinations to ‘compose’ and break down the \(Q\) area.

i) Overtur the model. Look at the sheet from the orange side: it identifies the vertex of the square \(Q\) coinciding with the vertex of one of the squares of area \(Q/16\). Bring this vertex to the center of the model.

j) What can you see now? With respect to what was observed in step (h), we note a green triangle: it is the composition of a square of area \(Q/16\) and two triangles each of value \(Q/32\) for a total green surface equal to \(Q/8\). We can therefore observe that the value of the green area is equal to the value of each of the orange rectangles now visible.

k) Considering the square of area \(Q/4\), adjacent to the cathetus of the large compound green triangle, retake the fold that defined its symmetry axis by identifying the two rectangles now visible. In this way a green trapeze is obtained as the sum of a rectangle of area \(Q/8\) and three triangles each of area \(Q/32\).

l) Repeat the previous step, thus repeating the fold that divides the two orange rectangles. You get, even in this case a green trapeze as the sum of other polygons. Looking at the model as a whole we note an orange square of area \(Q/4\), two green rectangles each of area \(Q/8\) and two green triangles, each of area \(Q/32\). Observing the green square, as the sum of the two triangles, bring its free vertex to the opposite one, with this vertex coinciding with one of the orange square.

m) Observing the model we note: the orange square of area \(Q/4\), two green rectangles, each of area \(Q/8\) and a green triangle of area \(Q/32\) as the sum of two smaller green triangles, each of area \(Q/64\). Now it is necessary to pay attention because this is the only passage that does not insist on folds already made and the thickness of the layers of paper already superimposed begins to create small discomforts during the folding phase. Considering the triangle of area \(Q/32\), you have to use as a fold line the line passing through the vertex in common between the two sides and with a direction parallel to its hypotenuse. To verify the correctness of this fold you have to observe that the mobile vertices of the aforementioned triangles will be tilted to coincide with the sides of the green rectangles.

n) Now the model has an orange pentagon (obtained as a subtraction of a triangle of area \(Q/64\) from square area \(Q/4\); triangle with right-angled vertex coinciding with a vertex of the square), two squares, each of area \(Q/16\), and three triangles each of area \(Q/32\).

o) Unfold the model and observe the pattern.

p) Look at the different kind of fold: mountain and valley. Use them to close the model by simultaneously pinching the orange rectangles bringing the surfaces in contact and bending them under the model (fig. 4).

q) Observe the model: one orange square area \(Q/4\), four little green triangles area \(Q/64\) each and two orange triangles area \(Q/32\) each. Compare triangles area: the value of the sum of each colour is the same, \(Q/16\).

r) At the end, it is possible to observe that folding square \(Q\) it is possible to find square \(A_0\) area \(Q/4\) and square \(B_0\) area \(A_0/4\) (Fig. 3).
3 THE HAND TOUCHES THE EVIDENCE AND THE MIND DISCOVERS THE SECRET

The use of origami models in primary school education, ranging from image to geometry, accompanies users to discovery in a privileged learning path. The origami is in fact offered as a geometry to be seen, touched and manipulated, in short a geometry to be discovered in an experiential way. The identified model, in the process optimized for its sharing, turned out to be the artifact of a 2D Fractal model. So, it allows to introduce not only the recognition of plane figures and the concept of passage of scale for which it was designed, but also to observe the fractions, to visualize equivalent areas and, finally, to become familiar with the concept of fractal. The following are observations on the finished model.

3.1 Recognizing polygons

In a similar way to what was observed during the folding phase (see 2.2.1.), on the finished model too it is possible to recognize polygons as such and polygons as sum or subtraction of several figures. In the final model of the fold sequence illustrated above we can note, on the main side, a square and four triangles, two of which perceptively render the idea of a square as adjacent with the hypotenuse and belonging to the same layer of paper. On the back, three squares belonging to different layers are visible, five triangles belonging to the same layer and combined so as to allow the perception of more squares as the sum of triangles and two other triangles, belonging to different layers. It is interesting to observe in Fig. 5 the outcome of the same sequence without reversing the support in step i of Fig. 3: we see two trapezes (absent in the previous model) and we see the square of area $Q/4$ only as a sum of other polygons. In this case, the color helps to perceive it as unitary.

3.2 Observing equivalent areas and fractions

The model turned out to be an excellent tool for observing equivalent areas not only in the folding phase (as observed in 2.2.1) but also in the finished model, playing between the different thicknesses of paper. In fig. 5 (bc) it is in fact possible to observe the front and the back of the same model and read the square orange $Q/4$ while, on the back, the same area $Q/4$ can be seen as an aggregation of parts: three squares with value $Q/16$, a triangle with value $Q/32$ and two triangles with value $Q/64$. The front (Fig. 5b) explains the proportional relationship between the two orange squares, one a quarter of the other, while the back (Fig. 5c) shows the equivalent of the orange square, its half and also its fourth: the visual communication in this case is less explicit because it cannot use the different layers of paper or the chromatic values highlights forms, while perimeters are defined by folds.

3.3 Understanding the scale of representation

The scaled representation is a task that makes use of both disciplinary approaches—math and drawing—and the proposed activity with an origami sheet is born with the intention of visualizing its geometric transformations in a dynamic way. The intrinsic complexity of the concept of scale of reduction is proper of its double dimensional value: the indication 1:2 refers to a linear dimension that has as a consequence an impact in its areal application. A segment of length $l$, on a 1:2 scale, is $l/2$ long and a square of edge $l$ and area $l^2$ will have, on a scale of 1:2, side $l/2$ and area $l^2/4$. Understanding it theoretically is much more complex than actually doing it or perceiving it visually.
3.4 Discovering fractals

"Between recognizing certain forms and producing them, there is a long distance, which the childish intelligence, attention, and memory must have traversed in its slow and gradual advance before the latter stage is reached". [7] Folding an origami is always a good opportunity to discover and develop geometric skills: each origami model allows to "see" multiple transformations in the folding sequence. Once the folding sequence is learned, this can be cyclically repeated (respecting the limits of the thickness of the paper layers and their size) giving life to a fractal model.

Starting from a square sheet of Area \( Q \), in the preliminary step \( n=0 \) the folding process leads to two squares of Area: \( A_0 = 1/4 Q \) and \( B_0 = 1/4 A_0 \). Repeating the folding process in square \( B_0 \) this again turns into two squares of Area: \( A_1 = 1/4 B_0 = 1/16 A_0 \) and \( B_1 = 1/4 A_1 \). At step \( n \) folding square \( B_{n-1} \) two squares are observed: \( A_n = 1/4 B_{n-1} = 1/16 A_{n-1} \) and \( B_n = 1/4 A_n \). Thus the geometric sequence \( A_n = [(1/16)^n] A_0 \) is produced (see Table 1).
<table>
<thead>
<tr>
<th>step</th>
<th>square</th>
<th>Folding Q to find $A_0$ and $B_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Q</td>
<td>$A_0 = \frac{1}{16}Q$, $B_0 = \frac{1}{16}A_0$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>step</th>
<th>Folding B₀ to find $A₁$ and $B₁$</th>
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<td>1</td>
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<table>
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<tr>
<th>step</th>
<th>Folding B₁ to find $A₂$ and $B₂$</th>
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<tbody>
<tr>
<td>2</td>
<td>$A₀$, $A₁ = \frac{1}{4}B₀$, $A₂ = \frac{1}{4}B₁$, $B₂ = \frac{1}{4}A₂$</td>
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<table>
<thead>
<tr>
<th>step</th>
<th>Folding B₂ to find $A₃$ and $B₃$</th>
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</thead>
<tbody>
<tr>
<td>3</td>
<td>$A₀$, $A₁ = \frac{1}{4}B₀$, $A₂ = \frac{1}{4}B₁$, $A₃ = \frac{1}{4}B₂$, $B₃ = \frac{1}{4}A₃$</td>
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<table>
<thead>
<tr>
<th>step</th>
<th>Folding Bₙ₋₁ to find $Aₙ$ and $Bₙ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>$A₀$, $A₁ = \frac{1}{4}B₀$, $A₂ = \frac{1}{4}B₁$, $Aₙ = \frac{1}{4}Bₙ₋₁$, $Bₙ = \frac{1}{4}Aₙ$</td>
</tr>
</tbody>
</table>

It follows an algorithmic fractal process provided that at the final step $n$ the square $B_n$ is excluded: folding, you have to hide the last square $B_n$ in the backside.

![Figure 8. Geometric sequence. On the Left: $A₀$, $A₁$, $A₂$, B₂. On the right: $A₀$, $A₁$, $A₂$.](image-url)

## 4 CONCLUSIONS

As a teacher of design I must emphasize that the problem of the representation scale is a problem related not only to the quantity of data to be represented but also, and above all, to the quality of the same data and, regarding this aspect, the projected origami experience is not effective. There is no doubt that in primary school it is fundamental to introduce first the dimensional aspects rather than the qualitative critical ones and therefore the experiences in the classes confirmed the effectiveness of the
activity. The conjugation of lessons on geometric language during the folding sequence and the subsequent consideration of what is observed in its transformations make the model interesting not only for primary school students but also for students of secondary education. Indeed, the model, designed with the intention of explaining the scale ratio 1:2, ranging from linear to spatial relationships, was later revealed as a tool to compare equivalent areas and also a useful tool for displaying $2^{\left(-2n\right)}$ geometric sequence. Although fractals are not included in the current ministerial guidelines for the primary school curricula, many mathematical topics would allow the description and use of an interdisciplinary approach - combining mathematics and representation and involving spatial intelligence as well as emotional intelligence - allows deepen them indirectly by offering useful bases for higher levels of schooling.

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